Begins with an algebra problem

Franklin

June 23, 2020

Versions of it

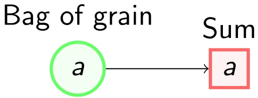
- proposed by Leo Moser in 1957, American Mathematical Monthly.
- 1962 Moscow Olympiad
- Used in 1991 in a student mathematical contest in St.Petersburg, USSR.
- Euclid 2020

The problem

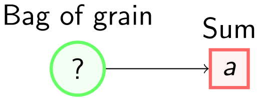
Problem

A malicious farmer's apprentice was asked to provide the list of weights of n bags of grain. Instead he weighed them two at a time and recorded all $\frac{n(n-1)}{2}$ combined weights written down in some random order. Is it possible to find the weights of bags (up to permutation of bags)?

Case of 1 number.

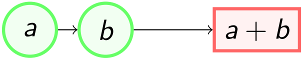


Case of 1 number. No problem



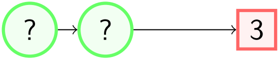
Case of 2 numbers.

Given their sum.



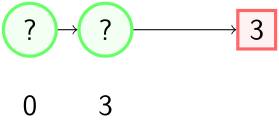
Case of 2 numbers.

Given their sum.



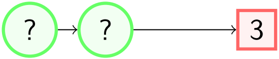
Case of 2 numbers.

Given their sum.



Case of 2 numbers.

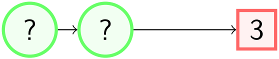
Given their sum.



0	3
1	2

Case of 2 numbers. Impossible

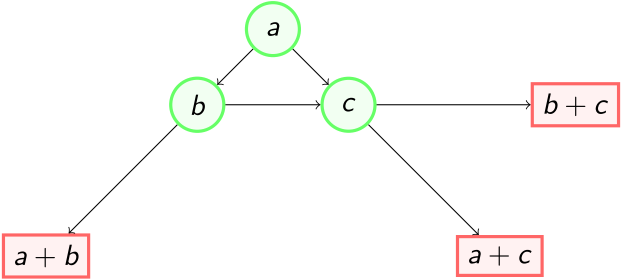
Given their sum.



0	3
1	2
-1	4
⋮	⋮

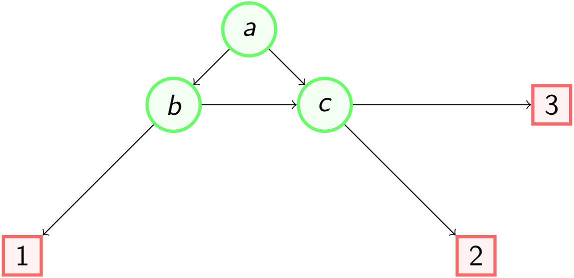
Case of 3 numbers

Given the sums of all pairs.



Case of 3 numbers

Given the sums of all pairs.



Case of 3 numbers

Adding all equations

$$\begin{array}{rr} a + b \qquad & = 1 \\ a \qquad + c & = 2 \\ b + c & = 3 \\ \hline 2(a + b + c) & = 6 \end{array}$$

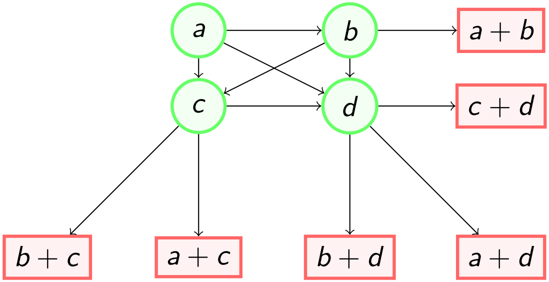
So, $a + b + c = 3$.
Since, $b + c = 3$, then $a = 0$.
Once we have a, the rest of the numbers are found.

$$b = 1, c = 2$$

Case of 4 numbers

$$a \leq b \leq c \leq d$$

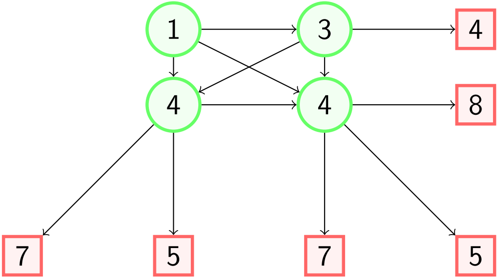
Given the sums of all pairs.



Case of 4 numbers

$$a \leq b \leq c \leq d$$

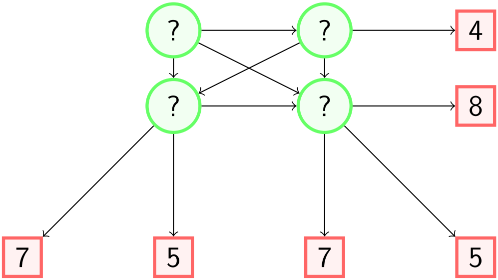
Given the sums of all pairs.



Case of 4 numbers

$$a \leq b \leq c \leq d$$

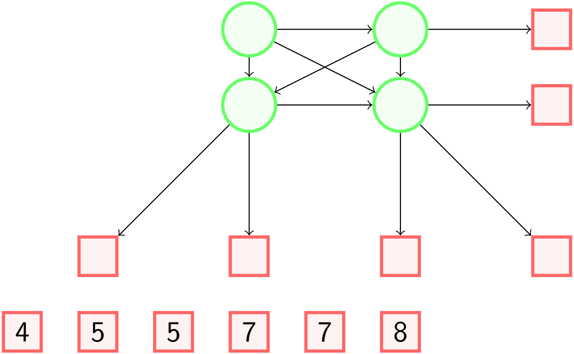
Given the sums of all pairs. Still possible.



Case of 4 numbers

$$a \leq b \leq c \leq d$$

Given the sums of all pairs. In no particular order.



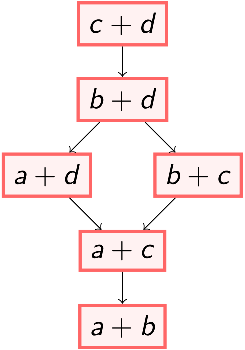
Case of 4 numbers

$$a \leq b \leq c \leq d$$

- $a + b$ is the smallest sum. $= 4$
- $a + c$ the second smallest. $= 5$
- $c + d$ the largest. $= 8$
- $b + d$ the second largest. $= 7$
- But $a + d$ and $b + c$ ... One is 5 and the other 7.

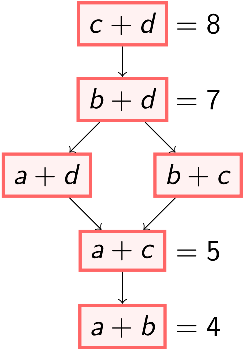
Case of 4 numbers

We can draw a graph:



Case of 4 numbers

We can draw a graph.



Case of 4 numbers

There are actually two solutions.

If $a + d = 5$ and $b + c = 7$	If $a + d = 7$ and $b + c = 5$
Then $(a, b, c, d) = (1, 3, 4, 4)$	Then $(a, b, c, d) = (2, 2, 3, 5)$

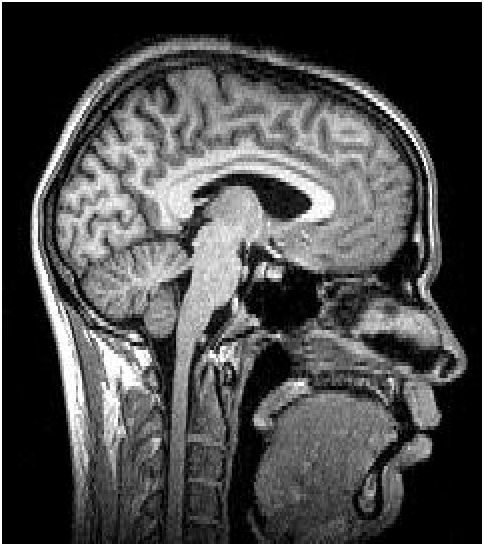
Therefore, the case of 4 numbers cannot be uniquely determined.

Is this all a game?

Is this all a game?

Well, yes ... but not only a game.

X-ray tomography



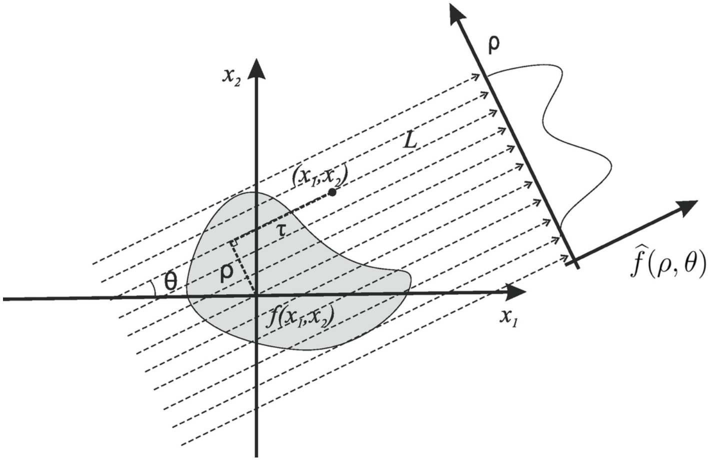
X-ray tomography

Computed tomography scan (CT Scan)

X-ray tomography

Radon transform

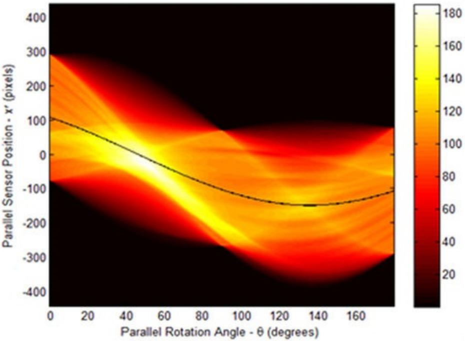
Sort of sums the density of an unknown object along each line.
The CT scan uses the sums to compute how the unknown object looks like.

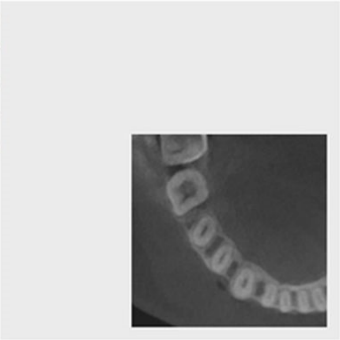


X-ray tomography

Sums

The original object.
Ribs ... I think.





X-ray tomography

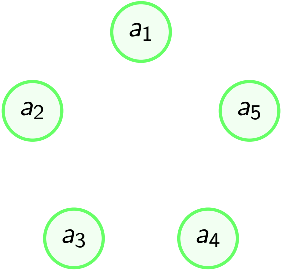
Our problem is a simplified version of a CT Scan of a collection of numbers.

$$\begin{cases} a \\ b \\ c \end{cases} \xrightarrow{\text{Tomography}} \begin{cases} s_1 \\ s_2 \\ s_3 \end{cases} = \begin{cases} a+b \\ a+c \\ b+c \end{cases}$$

$$\begin{cases} s_1 \\ s_2 \\ s_3 \end{cases} \xrightarrow{\text{Reconstruction}} \begin{cases} a &= \frac{s_1+s_2-s_3}{2} \\ b &= \frac{s_1-s_2+s_3}{2} \\ c &= \frac{s_2+s_3-s_1}{2} \end{cases}$$

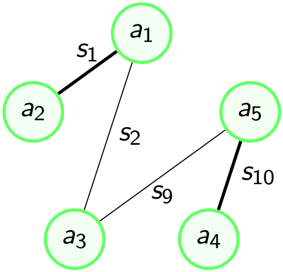
Case of 5 numbers

We are given the 10 sums of all pairs $s_1 \leq s_2 \leq \ldots \leq s_9 \leq s_{10}$.



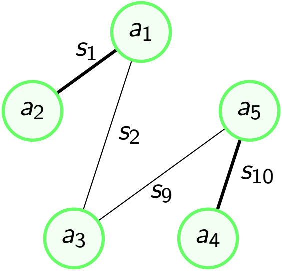
Case of 5 numbers

We are given the 10 sums of all pairs $s_1 \leq s_2 \leq \ldots \leq s_9 \leq s_{10}$.



Case of 5 numbers

We are given the 10 sums of all pairs $s_1 \leq s_2 \leq \ldots \leq s_9 \leq s_{10}$.



$4(a_1 + a_2 + a_3 + a_4 + a_5) = s_1 + s_2 + \ldots + s_{10}$ is known.
So, we can solve for a_3!

$$a_3 = (a_1 + a_2 + a_3 + a_4 + a_5) - s_1 - s_{10} = \frac{s_1 + s_2 + \ldots + s_{10}}{4} - s_1 - s_{10}$$

Another solution for the case of 3 numbers

Symmetric polynomials. Power-sum symmetric polynomials.

$$\mathbf{s_1} + \mathbf{s_2} + \mathbf{s_3} = (3 - 2^0)(a_1 + a_2 + a_3)$$
$$\mathbf{s_1^2} + \mathbf{s_2^2} + \mathbf{s_3^2} = (3 - 2^{2-1})(a_1^2 + a_2^2 + a_3^2) + (a_1 + a_2 + a_3)^2$$
$$\mathbf{s_1^3} + \mathbf{s_2^3} + \mathbf{s_3^3} = (3 - 2^{3-1})(a_1^3 + a_2^3 + a_3^3) + 3(a_1 + a_2 + a_3)(a_1^2 + a_2^2 + a_3^2)$$

Another solution for the case of 3 numbers

Newton formulas

$$
\begin{aligned}
a_1 + a_2 + a_3 &= \mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3} \\
a_1^2 + a_2^2 + a_3^2 &= (a_1 + a_2 + a_3)^2 - 2(\mathbf{a_1 a_2} + \mathbf{a_1 a_3} + \mathbf{a_2 a_3}) \\
a_1^3 + a_2^3 + a_3^2 &= (a_1 + a_2 + a_3)^3 - 3(a_1 + a_2 + a_3)(a_1 a_2 + a_1 a_3 + a_2 a_3) \\
&\quad + 3\mathbf{a_1 a_2 a_3}
\end{aligned}
$$

Another solution for the case of 3 numbers

Vieta's formulas and Elementary symmetric polynomials

Theorem

If

$$e_1 = a_1 + a_2 + a_3$$
$$e_2 = a_1 a_2 + a_1 a_3 + a_2 a_3$$
$$e_3 = a_1 a_2 a_3$$

then

$$x^3 - e_1 x^2 + e_2 x - e_3 = (x - a_1)(x - a_2)(x - a_3)$$

Therefore, a_1, a_2, a_3 are the solutions of the equation

$$x^3 - e_1 x^2 + e_2 x - e_3 = 0$$

Case of n numbers

Theorem

If n is not a power of 2 and we are given the $m = \frac{n(n-1)}{2}$ sums $s_1, s_2, ..., s_m$ of all the possible pairs of n numbers $a_1, a_2, ..., a_n$, then we can recover the values of $a_1, a_2, ..., a_n$.

Case of n being a power of 2

Theorem

Not possible to recover the numbers

$$a_1, a_2, \ldots, a_n$$

from their pairwise sums

$$s_1, s_2, \ldots, s_{\frac{n(n-1)}{2}}$$

Thue-Morse sequence

0

$0, 1$

$0, 1, 1, 0$

$0, 1, 1, 0, 1, 0, 0, 1$

$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0$

Copy previous block and repeat it changing $1 \leftrightarrow 0$

$\ldots$

Thue-Morse sequence

$$\begin{aligned} \text{Natural numbers } &\to 1, 2, 3, 4, 5, 6, 7, 8 \\ \text{Morse sequence } &\to 0, 1, 1, 0, 1, 0, 0, 1 \end{aligned}$$

Put the numbers with a 0 below in a group and the ones with a 1 in another group.

$$A = \{1, 4, 6, 7\}, \qquad B = \{2, 3, 5, 8\}$$

These two groups give the same collections of sums of all its pairs.

$$\text{Sums of pairs: } 5, 7, 8, 10, 11, 13$$

Thue-Morse sequence

$$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$$

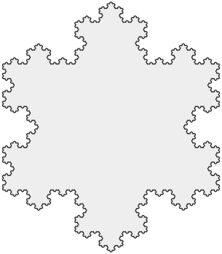
- Never periodic.
- There is never a block X that repeats three times XXX.
- The portion obtained after $2k$ steps is a palindrome.

Koch snowflake

The Thue-Morse sequence gives the instructions to draw a fractal.

$$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$$

- If 0, then move ahead $1cm$
- If 1, then rotate 60°



Generalizations of the problem

Problem

A malicious farmer's apprentice was asked to provide the list of weights of n bags of grain. Instead he weighed them two at a time and recorded all $\frac{n(n-1)}{2}$ combined weights written down in some random order. Is it possible to find the weights of bags (up to permutation of bags)?

What if we sum every three bags? Or every 4?
Every group of size s?

Generalizations of the problem

Number theory starts coming into the mix.

Definition (Moser polynomials)

$$F_{s,k}(n) = \sum_{p=1}^{s} (-1)^{p-1} p^{k-1} \binom{n}{s-p}$$

Example ($s = 2$)

$$F_{2,k}(n) = n - 2^{k-1}$$

Example ($s = 3$)

$$2F_{3,k}(n) = n^2 - n(2^k + 1) + 2 \cdot 3^{k-1}$$

Generalizations of the problem

For what positive integer k does it exist a positive integer n such that

$$n^2 - n(2^k + 1) + 2 \cdot 3^{k-1} = 0$$

Example

$$F_{3,3}(n) = \frac{1}{2}(n-3)(n-6)$$

Therefore 6 numbers probably cannot be determined by the sums of every one of its triples. Actually the sets below give the same sums of triples

$$A = \{0, 5, 9, 10, 11, 13\}; B = \{1, 5, 8, 9, 10, 15\}$$

$$C = \{1, 6, 7, 8, 11, 15\}; D = \{3, 5, 6, 7, 11, 16\}$$

Questions

June 23, 2020